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Inertial effects on clusters of spheres falling in a viscous fluid

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A compact cluster of 3 to 6 rigid equal spheres is falling under gravity in a viscous liquid. The small effects of inertia on a horizontal regular polygonal configuration are that the polygon expands as it falls and small perturbations from this configuration die out, although when the polygon is large enough it becomes weakly unstable. This is an extension of the analysis of Hocking (1964), which was applied to the experiments of Jayaweera, Mason & Slack (1964).

1. Introduction

Jayaweera *et al.* (1964) observed compact clusters of equal spheres falling in a viscous liquid. If the number n of spheres in the cluster lies between 3 and 6 inclusive, and if the Reynolds number Re based on radius and velocity of fall is between 0·06 and 7·0, the spheres form a regular polygon in a horizontal plane, which expands slowly at a decreasing rate as they fall. Oscillations about this polygonal configuration decay at the same time. If $Re < 0{\cdot}06$, or if the spheres are too widely separated, there is no tendency to form a regular arrangement, and if $Re > 7$ the spheres simply separate. If n exceeds 6 the polygonal arrangement is apparently unstable.

Hocking (1964), among other things, showed on the basis of Stokes equations that if the separation of the spheres is large compared to their radii, such a polygonal configuration of 3–6 spheres is stable to small perturbations, but for 7–12 spheres it is unstable. The polygon should remain of constant size during the fall, and the amplitude of small oscillations of the spheres about their mean relative positions is also constant. He also speculated that the expansion of the polygon and the damping of the oscillations should be ascribed to inertial effects ignored in this theory.

The objective of this paper is to investigate this speculation, on the basis of an extension of his analysis valid when the Reynolds number Re for each individual sphere is small but not zero. Each sphere falls vertically through the liquid with an appropriate terminal velocity, but also moves with the average velocity of the fluid in its neighbourhood. This velocity arises from the flow round the other distant spheres, and gives rise to slow relative motions. The innovation here is in using the velocity field appropriate for Oseen flow rather than the Stokes flow, and in the modifications to the normal modes in the stability analysis which allow for the change of the equilibrium configuration with time. Specific results are:

(i) On the basis of the Stokes equations, $Re = 0$, the polygon will not expand, nor will small oscillations be damped, even if the radius of the spheres is comparable with their separation.

(ii) When $0 < Re \ll 1$ and the spheres are widely spaced, the polygon expands with time at a rate which ultimately decreases to zero.

(iii) When $0 < Re \ll 1$ and the spheres are widely but not too widely spaced, the inertial correction to Hocking's analysis is relatively small, but for $3 \leqslant n \leqslant 6$ small oscillations are damped. Under these conditions both the rate of expansion of the polygon and the rate of decay of the oscillations are proportional to Re, but from numerical results it appears that for the amplitude of the oscillations to be halved, the radius of the polygon has to increase by a factor which depends on the mode of oscillation but is at least 16.

(iv) For any Re, if the spheres are sufficiently far apart, the polygonal configuration is unstable, though the interactions are extremely weak, and this effect is probably unobservable.

The polygon formation described by Jayaweera *et al.* (1964), took place mainly at larger Reynolds numbers and smaller separations than those within the purview of this theory. However, the qualitative agreement over the later stages of polygon formation is satisfactory, although the theoretical value for the damping coefficients seems too small.

The equations of motion used by Hocking (1964, equation (4)) are valid asymptotically in the limit $a/s \to 0$, $Re\, s/a \to 0$, where s is a typical distance between the spheres. For given small Re this fails if s/a becomes too large; but if $Re\, s/a$ is still small but not negligible we may easily compute the first correction to Hocking's equations in powers of the Reynolds number.

If, on the other hand, $Re\, s/a \gg 1$, Hocking's equations are not even a valid first approximation. We may, however, easily use the theory of the Oseen region round an isolated sphere to obtain similar expressions for the relative motions. These are discussed briefly in § 3.

2. The case $Re\, s/a \ll 1$

If $\boldsymbol{\rho}_j$ is the position of a point P relative to the jth sphere, the fluid velocity at P relative to a frame of reference moving with the Stokes terminal velocity $-U\mathbf{z}$ of each of the spheres is

$$\begin{aligned}\mathbf{u} = -U\mathbf{z} + U\sum_j \tfrac{3}{4}a\left\{\frac{1}{\rho_j}\mathbf{z} + \frac{1}{\rho_j^3}(\mathbf{z}\cdot\boldsymbol{\rho}_j)\,\boldsymbol{\rho}_j\right\} \\ + U\sum_j \tfrac{3}{16}Re\left\{-\frac{2}{\rho_j}(\mathbf{z}\cdot\boldsymbol{\rho}_j)\,\mathbf{z} + \frac{1}{\rho_j}\left(1 - \frac{1}{\rho_j^2}(\mathbf{z}\cdot\boldsymbol{\rho}_j)^2\right)\boldsymbol{\rho}_j\right\} \\ + O\left(U\left(\frac{a}{\rho_j}\right)^3\right) + O\left(U\frac{a^2}{\rho_j^2}\right) + O\left(U\,Re^2\frac{\rho_j}{a}\right) + O\left(U\,Re\frac{a}{s}\right).\end{aligned} \quad (1)$$

In this formula $\mathbf{z}$ is the unit vector parallel to the downward vertical. The second term is the superposition of the disturbance velocities from the Stokes flow of a uniform stream past n *isolated* spheres, only the dominant part at large distances (the Stokeslets) being retained. It is of order Ua/ρ_j and in this form is only appropriate if $a/\rho_j \ll 1$ for all ρ_j, the error being $O(a/\rho)^3$. The third term is of order Re, is independent of ρ_j, and is the dominant part of the superposition of the first corrections to these Stokes flows due to the non-linear acceleration terms $Re\,(\mathbf{u}\,.\,\nabla)\,\mathbf{u}$ in the Navier–Stokes equations.

The use of the Stokes flows round isolated spheres is valid because the interaction velocity field due to the presence of one in the perturbation due to another is of order $(a/s)\,(a/\rho_j)$. Furthermore, the motion of the spheres relative to our co-ordinate frame is $O(Ua/s)$, so the local rate of change of the fluid velocity $\mathbf{u}$ is of order $U(a/s^2)\mathbf{u}$, whereas the non-linear acceleration terms retained here are $O(U/s)\,\mathbf{u}$. Thus it is justified to regard the flow as instantaneously steady and to use known results from the non-linear steady flow past isolated spheres.

The first inertial correction in the Stokes region round an isolated sphere is given by Proudman & Pearson (1957, equation (3.42)); of this only the part of order $Re\,(a/\rho_j)^0$ is retained. In their derivation an arbitrary additive velocity field describing the Stokes flow of a uniform stream round the sphere appears in this correction. The constant involved was determined by matching to the outer, Oseen, region where $Re\,\rho_j/a$ is of order unity or larger. Here the force on the sphere is given (expressed as its Stokes terminal velocity U), rather than the velocity relative to the fluid at infinity. Thus the arbitrary constant must be chosen so that the additional force of order Re vanishes. This condition is incorporated in equation (1). From the solution in the Oseen region, however, in which the flow in the neighbourhood of the whole cluster of spheres appears as a uniform stream $-U\mathbf{z}$ and a *single* Stokeslet point singularity of strength proportional to the force on the whole cluster, we see that the fluid in the region $a/\rho_j \ll 1$, $Re\,\rho_j/a \ll 1$ is moving as a whole relative to the fluid at infinity with velocity $-\frac{3}{8}n\,Re\,U\mathbf{z}$. This means that our co-ordinate frame is falling at a speed $U(1-\frac{3}{8}n\,Re)$ relative to the fluid at infinity, while the centre of mass of the cluster is falling at a rate of order Ua/s relative to the co-ordinate frame. For computing relative velocities within the cluster, equation (1) is appropriate as it stands.

The acceleration terms associated with the time dependence of the basic flow due to relative motions of the spheres, and with the interaction velocity field due to the presence of one in the perturbation due to another, give rise to a correction which is smaller by $O(a/s)$. Indeed, the third term of equation (1) arises solely from convection of the individual Stokeslet fields by the uniform stream. As for given Re and s, P becomes more distant from the cluster, the third term of equation (1) becomes comparable with the second and the basis of the calculation fails. Instead the full theory of the Oseen region is required.

If $Re \ll a/s \ll 1$, the departures of the spheres from steady fall may be calculated to a first approximation in the manner already indicated by Hocking (1964). Now, however, the second and third terms of equation (1) are used, instead of merely the second. If the term $j = i$ in the summation is omitted they give the velocity in the neighbourhood of the ith sphere due to the presence of the others, and, in the absence of additional restraining forces, or substantial accelerations, the ith sphere moves relative to our co-ordinate frame with this velocity. The rate of change of the flow pattern is so small that the inertia of the spheres may be neglected, together with the inertia of the fluid within a distance of order a from them. With a length scale s_0, a typical initial distance between spheres, and a time scale $\frac{4}{3}s_0^2/aU$, we have in dimensionless form

$$\frac{d\mathbf{r}_i}{dt} = \sum_{j\neq i}\left\{\frac{1}{r_{ij}}\,\mathbf{z}+\frac{1}{r_{ij}^3}\,(\mathbf{z}.\mathbf{r}_{ij})\,\mathbf{r}_{ij}\right\} + \frac{1}{4}\frac{Re\,s_0}{a}\sum_{j\neq i}\left\{-\frac{2}{r_{ij}}\,(\mathbf{z}.\mathbf{r}_{ij})\,\mathbf{z}+\frac{1}{r_{ij}}\left(1-\frac{1}{r_{ij}^2}\,(\mathbf{z}.\mathbf{r}_{ij})^2\right)\mathbf{r}_{ij}\right\}. \tag{2}$$

Here, as in Hocking's equation (4), $\mathbf{r}_i$ is the dimensionless position vector of the centre of the ith sphere and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. The inclusion of the second, correction, term in this equation is only valid if it is small compared to the first, and only sensible if it is large compared to other terms omitted. This condition may be summed up as

$$(a/s)^2 \ll Re \ll a/s.$$

The Stokeslet velocity field due to a single downward moving sphere is symmetrical about the horizontal plane through the sphere, decreasing with distance from it, and everywhere downwards. It is sketched in figure 1 (a). If a set of n spheres lie in a horizontal plane the effect of the jth sphere will be to give the ith one an additional downward velocity. Hence the rate of fall of the whole cluster is increased by interactions. If the ith sphere is now displaced slightly vertically downwards, its downward velocity is to first order unaffected but it tends to move horizontally. After a time this affects the average value of the reciprocal of its distance from the others and its vertical velocity changes. Whether the net effect is to tend to restore all the particles to the same plane or to increase relative vertical displacements has to be found from the detailed calculations of Hocking, but because of the symmetrical nature of the Stokeslet field vertical displacements are associated directly with horizontal velocities and vertical accelerations. The stability calculation thus involves only the square of the frequency of an oscillation, and this frequency is either real or pure imaginary. The oscillations are thus essentially undamped.

This situation is an essential consequence only of the symmetry properties of the Stokes equations and the assumption of small perturbations about a configuration which is symmetrical about a horizontal plane. It does not depend on the assumption $a/s \ll 1$. The mirror image–time reversal theorem of Bretherton (1962) shows that each vertical perturbation velocity $d(\mathbf{r}_i.\mathbf{z})/dt$ is an even function of the set of relative vertical displacements $(\mathbf{r}_i - \mathbf{r}_j).\mathbf{z}$, whereas the horizontal velocities are odd functions of this set, for if any configuration is replaced by its mirror image in a horizontal plane the vertical velocities are unaltered and the horizontal velocities reversed. But, by symmetry, a regular polygon is a possible equilibrium configuration in which all relative velocities vanish. For small perturbations the relative vertical velocities may be expressed to first order in terms of the horizontal displacements only, and the horizontal velocities in terms of the vertical displacements even if a/s is of order unity. Thus within the context of Stokes equations there is no room for damped oscillations, and to explain the observation recourse must be had to inertial effects (conclusion (i)).

The inertial correction described by a single term of the second summation in equation (1) has a quite different form (figure 1 (*b*)). It is antisymmetric about a horizontal plane through the sphere, with outflow in this plane and in flow along the vertical axis. A plane regular polygon will thus expand due to this term, the relative velocity of each sphere being horizontal, independent of the size of the polygon, and equal to $\frac{3}{16} Re\, U$ times the vector sum of unit vectors directed from all the other spheres. The constant rate of expansion will continue until $Re\, s/a$ is of order unity and equation (2) becomes invalid.

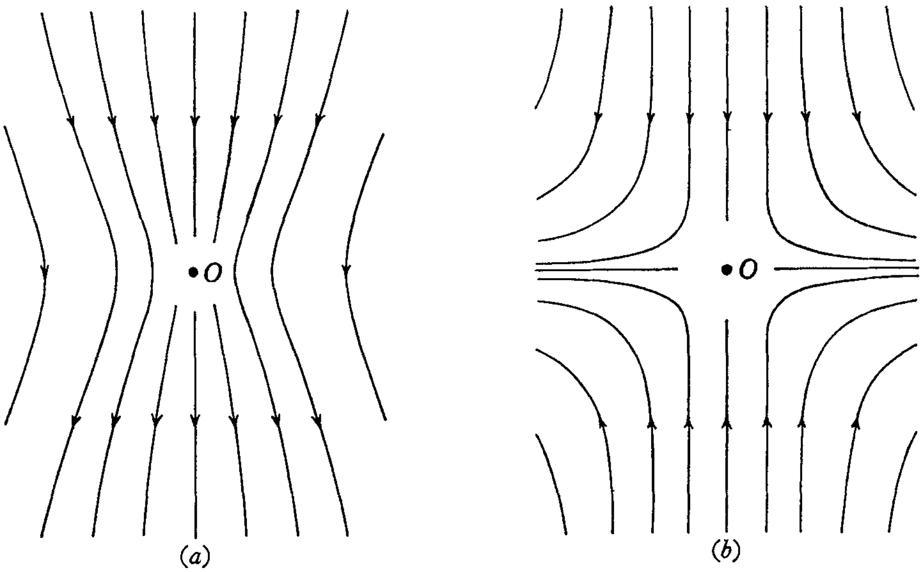

FIGURE 1. (*a*) Streamlines of perturbation flow due to small downward moving sphere at the origin (Stokeslet). (*b*) First inertial correction in region $Re s/a \ll 1$.

The effect of the inertial correction on the small oscillations about a regular polygon described by Hocking for $Re = 0$ requires more detailed analysis. As long as $Re\, s/a$ is small, the correction is small and equation (2) may be used, but in the later stages of the expansion different considerations apply. The first is the more interesting case, and will be dealt with first. Two things may be said at once. The frequency of the oscillations is proportional to s^{-2}, which decreases as the polygon expands. However, a large number, $O(Re\, s/a)^{-1}$, of such oscillations occur before the frequency has changed substantially, so conditions may be described as 'slowly varying'. Secondly, the additional term gives rise to vertical velocities proportional to small vertical displacements and horizontal velocities proportional to horizontal perturbations. This type of term acts as a small positive or negative damping, and the amplitude of the oscillations may be expected to change slowly with time. The effect over one period is essentially small for as long as equation (2) is valid, but the cumulative change over many periods may be large.

Following Hocking, we consider small perturbations (x_i, y_i, z_i) of the ith sphere from its mean relative position

$$\bar{\mathbf{r}}_i = \sigma(\epsilon t)\{\mathbf{x}\cos[2\pi(i/n)] + \mathbf{y}\sin[2\pi(i/n)]\} \quad (i = 1, \ldots, n),$$

where $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, horizontal unit vectors. $\sigma(\epsilon t)$ is a measure of the dimensionless size of the polygon relative to its initial size s_0, and is given by

$$\sigma(\epsilon t) = 1 + \tfrac{1}{4}(\cot \pi/2n)\,\epsilon t,$$

where, for brevity, we have written $\epsilon = Re\, s_0/a$. If we denote the n-dimensional vectors $\{\mathbf{x}_i\}, \{\mathbf{y}_i\}, \{\mathbf{z}_i\}$ $(i = 1, \ldots, n)$ by $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}$ respectively, we obtain linearized equations for the perturbations of the type

$$\left.\begin{aligned} d\boldsymbol{\xi}/dt &= -\sigma^{-2}\mathbf{B}.\boldsymbol{\zeta} - (\epsilon/\sigma)(\mathbf{M}.\boldsymbol{\xi} - \mathbf{N}.\boldsymbol{\eta}), \\ d\boldsymbol{\eta}/dt &= -\sigma^{-2}\mathbf{C}.\boldsymbol{\zeta} - (\epsilon/\sigma)(-\mathbf{N}.\boldsymbol{\xi} + \mathbf{L}.\boldsymbol{\eta}), \\ d\boldsymbol{\zeta}/dt &= \sigma^{-2}(\mathbf{B}.\boldsymbol{\xi} + \mathbf{C}.\boldsymbol{\eta}) + (\epsilon/\sigma)(\mathbf{L} + \mathbf{M}).\boldsymbol{\zeta}. \end{aligned}\right\} \quad (3)$$

The coefficients B_{ij}, etc., of the $n \times n$ matrices $\mathbf{B}$, etc., are given for $i \neq j$ by

$$B_{ij} = \frac{\sigma^2}{(\bar{r}_{ij})^3}(\bar{\mathbf{r}}_{ij}.\mathbf{x}) = -\frac{\sin\{\pi(i+j)/n\}}{4\sin\{\pi(i-j)/n\}\,|\sin\{\pi(i-j)/n\}|},$$

$$C_{ij} = \frac{\sigma^2}{(\bar{r}_{ij})^3}(\bar{\mathbf{r}}_{ij}.\mathbf{y}) = +\frac{\cos\{\pi(i+j)/n\}}{4\sin\{\pi(i+j)/n\}\,|\sin\{\pi(i-j)/n\}|},$$

$$L_{ij} = \frac{\sigma}{4(\bar{\mathbf{r}}_{ij})^3}(\bar{\mathbf{r}}_{ij}.\mathbf{x})^2 = \frac{\sin^2\{\pi(i+j)/n\}}{8\,|\sin\{\pi(i-j)/n\}|},$$

$$M_{ij} = \frac{\sigma}{4(\bar{\mathbf{r}}_{ij})^3}(\bar{\mathbf{r}}_{ij}.\mathbf{y})^2 = \frac{\cos^2\{\pi(i+j)/n\}}{8\,|\sin\{\pi(i-j)/n\}|},$$

$$N_{ij} = \frac{\rho}{4(\bar{r}_{ij})^3}(\bar{\mathbf{r}}_{ij}.\mathbf{x})(\bar{\mathbf{r}}_{ij}.\mathbf{y}) = -\frac{\sin\{\pi(i+j)/n\}\cos\{\pi(i+j)/n\}}{8\,|\sin\{\pi(i-j)/n\}|},$$

and

$$B_{ii} = -\sum_{j \neq i} B_{ij}, \quad C_{ii} = -\sum_{j \neq i} C_{ij}, \quad \text{etc.}$$

Hocking found the normal mode solutions of equation (3) for the case $\epsilon = 0$, $\sigma = 1$, as follows. If $\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi}$ are given by the real parts of

$$\mathbf{X}_r e^{i\mu_r t}, \quad \mathbf{Y}_r e^{i\mu_r t}, \quad \mathbf{Z}_r e^{i\mu_r t},$$

then μ_r^2 must be a latent root of the matrix

$$-\tfrac{1}{4}\mathbf{A} = \mathbf{B}.\mathbf{B} + \mathbf{C}.\mathbf{C},$$

corresponding to the eigen-vector $\mathbf{Z}_r$. Then

$$\mathbf{X}_r = -(1/i\mu_r)\,\mathbf{B}.\mathbf{Z}_r, \quad \mathbf{Y}_r = -(1/i\mu_r)\,\mathbf{C}.\mathbf{Z}_r. \quad (4)$$

Now $\mathbf{A}$ is a circulant matrix (i.e. $A_{i+1,j+1} = A_{ij}$) and is also real and symmetric ($A_{ij} = +A_{ij}$). These properties arise from the invariance of the polygon to rotation through any multiple of $2\pi/n$ and to reflexion in the Ox axis. A great simplification in this problem arises because every circulant matrix has the same complete set of orthogonal eigen-vectors

$$\mathbf{Z}_r = (\omega_r, \omega_r^2, \ldots, \omega_r^n),$$

where ω_r is an nth root of unity

$$\omega_r = e^{2\pi i r/n} \quad (r = 1, \ldots, n).$$

Then

$$\mu_r^2 = \sum_{j=1}^{n} -\tfrac{1}{4}A_{nj}\cos\frac{2\pi rj}{n}.$$

Hocking showed that if $3 \leqslant n \leqslant 6$, μ_r^2 is positive for all r, except $r = 0$, for which it vanishes, but if $7 \leqslant n \leqslant 12$ at least one negative latent root exists. We shall confine our attention to the former case, in which the polygonal configuration is stable.

The coefficients in equation (3) are functions of time, so a straightforward normal-mode analysis is not available to us. The method used instead is based on standard techniques for the asymptotic solution of equations with almost periodic solutions, and is related to perturbation analysis in quantum theory.

We look for solutions of the form

$$\boldsymbol{\zeta} = \mathscr{R}\left[a(\epsilon t)\,\{\mathbf{Z}_r + \epsilon\sigma(\epsilon t)\,\mathbf{Z}_r' + O(\epsilon^2)\}\exp i\left\{\mu_r\int_0^t \frac{dt}{\sigma^2}\right\}\right] \quad (r = 1, \ldots, n-1), \tag{5}$$

with corresponding expressions for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. $a(\epsilon t)$ is a complex amplitude function which varies only slightly over the period $2\pi\sigma^2/\mu_r$ of an oscillation. The phase of the oscillation is given by

$$\mu_r\int_0^t dt/\sigma^2,$$

rather than by $\mu_r t$ and the direction of $\boldsymbol{\zeta}$ is not quite that of $\mathbf{Z}_r$, there being a small correction $\epsilon\sigma\mathbf{Z}_r'$. Higher-order corrections are of smaller order still for as long as $\epsilon\sigma \ll 1$. It is only under this condition that equation (3) is valid anyhow, but the time involved before it is violated embraces many oscillations and significant damping. We now substitute into equation (3) and formally equate to zero coefficients of successive powers of ϵ.

The zero-order equations reduce at once to the eigen-value problem solved by Hocking. The first-order equations are

$$i\mu_r\mathbf{X}_r' + \mathbf{B}.\mathbf{Z}_r' = -(\dot{a}/a)\,\sigma\mathbf{X}_r - (\mathbf{M}.\mathbf{X}_r - \mathbf{N}.\mathbf{Y}_r),$$
$$i\mu_r\mathbf{Y}_r' + \mathbf{C}.\mathbf{Z}_r' = -(\dot{a}/a)\,\sigma\mathbf{Y}_r - (-\mathbf{N}.\mathbf{X}_r + \mathbf{L}.\mathbf{Y}_r),$$
$$i\mu_r\mathbf{Z}_r' - (\mathbf{B}.\mathbf{X}_r' + \mathbf{C}.\mathbf{Y}_r') = -(\dot{a}/a)\,\sigma\mathbf{Z}_r + (\mathbf{L}+\mathbf{M}).\mathbf{Z}_r.$$

Here $\dot{a}$ is the derivative of a with respect to its argument (ϵt). On elimination of $\mathbf{X}_r'$ and $\mathbf{Y}_r'$ between these equations, and using the relations (4), we have

$$\{\mathbf{B}.\mathbf{B} + \mathbf{C}.\mathbf{C} - \mu_r^2\mathbf{I}\}.\mathbf{Z}_r' = -2i\mu_r(\dot{a}\sigma/a)\,\mathbf{Z}_r + i\mu_r\{\mathbf{L} + \mathbf{M} - \mu_r^{-2}\mathbf{P}\}.\mathbf{Z}_r, \tag{6}$$

where $\mathbf{P}$ is the matrix

$$\mathbf{B}.\mathbf{M}.\mathbf{B} + \mathbf{C}.\mathbf{L}.\mathbf{C} - \mathbf{B}.\mathbf{N}.\mathbf{C} - \mathbf{C}.\mathbf{N}.\mathbf{B}.$$

Equation (6) determines whether a solution of the assumed form exists. The matrix

$$-\tfrac{1}{4}\mathbf{A} - \mu_r^2\mathbf{I}$$

multiplying $\mathbf{Z}_r'$ is singular, precisely because μ_r^2 is a latent root of $-\tfrac{1}{4}\mathbf{A}$, so there is a solution only if the right-hand side is specially chosen. In this problem this condition may be obtained elegantly by noting that by symmetry

$$\mathbf{L} + \mathbf{M} - \mu_r^{-2}\mathbf{P}$$

is a circulant matrix, so

$$(\mathbf{L} + \mathbf{M} - \mu_r^{-2}\mathbf{P}).\mathbf{Z}_r = 2\kappa_r\mathbf{Z}_r$$

for some latent root $2\kappa_r$, and

$$\{\mathbf{B}.\mathbf{B}+\mathbf{C}.\mathbf{C}-\mu_r^2\mathbf{I}\}.\mathbf{Z}_q = (\mu_q^2-\mu_r^2)\,\mathbf{Z}_q \quad (q = 1, \ldots, n).$$

Hence if $\mathbf{Z}_r'$ is expanded as a linear combination of the $\mathbf{Z}_q$, equation (6) tells us that the coefficient of $\mathbf{Z}_q$ vanishes for all q for which $\mu_q^2 \neq \mu_r^2$ and that a solution exists at all if and only if

$$2i\mu_r\{-(\dot{a}/a)\,\sigma+\kappa_r\} = 0.$$

Then $\mathbf{Z}_r'$ is any linear combination of the eigenvectors of $-\frac{1}{4}\mathbf{A}$ associated with latent roots equal to μ_r^2. This indeterminacy may be resolved by considering terms of $O(\epsilon^2)$.

The two modes for which $r = n$ and $\mu_r^2 = 0$ are respectively associated with a uniform downward displacement and a uniform expansion of the polygon as a whole. The effect of the inertial correction for these is trivial. For $r = 1, \ldots, n-1$, to each normal mode of the problem with $\epsilon = 0$, there exists a modified solution with amplitude which varies slowly with time according to

$$a = a_0 \exp\left(\kappa_r \int_0^t \frac{d(\epsilon t)}{\sigma(\epsilon t)}\right) = a_0(\sigma)^{4\kappa_r \tan(\pi/2n)}. \tag{7}$$

This set of modified normal modes is complete, and any initial disturbance may be described as a superposition of oscillations, damped or growing according to equation (7). Values of $4\kappa_r \tan(\pi/2n)$ were obtained for $n = 3, \ldots, 6$, using the

n	r	μ_r^2	κ_r	$4\kappa_r \tan(\pi/2n)$
3	1	0·17	−0·11	−0·25
	2	0·17	−0·11	−0·25
4	1	0·35	−0·12	−0·21
	2	0·71	−0·18	−0·29
	3	0·35	−0·12	−0·21
5	1	0·40	−0·15	−0·19
	2	1·34	−0·19	−0·24
	3	1·34	−0·19	−0·24
	4	0·40	−0·15	−0·19
6	1	0·14	−0·17	−0·18
	2	1·58	−0·11	−0·12
	3	2·89	−0·25	−0·27
	4	1·58	−0·11	−0·12
	5	0·14	−0·17	−0·18

TABLE 1.

computer EDSAC 2. They are shown in table 1. The unrounded values confirmed a computation by hand that for $n = 3$, $\mu_2^2 = \mu_3^2 = \frac{1}{6}$ and $\kappa_2 = \kappa_3 = -\sqrt{3}/16$. Values of $4\kappa_r \tan(\pi/2n)$ are all negative and substantially smaller than unity, indicating weakly damped oscillations. For $n = 3$ the size of the polygon must increase sixteen times for the amplitude of the oscillations to decrease by a factor of two. For $n = 7$, on the other hand, for two modes μ_r^2 is negative and perturbations grow exponentially, and for two other modes κ_r is positive, indicating growing vibrations. We thus conclude that a configuration initially not too far removed from a horizontal regular polygon $Re\, s_0/a \ll 1$ and $3 \leqslant n \leqslant 6$ will relax slowly but systematically into that shape as it expands, but if $7 \leqslant n \leqslant 12$ this will not occur (conclusion (iii)).

3. The case $Re\, s/a \geqslant 1$

After sufficient time $Re\, s/a$ becomes comparable with unity and equation (3) no longer holds. Some conclusions may be quickly reached from the theory of the Oseen flow round an isolated sphere. If Re is small, the condition implies that the separation of the spheres is large compared to their radii, and that their relative motions are small compared to their speed of fall. A first approximation is again that each sphere tends to move relative to the others with a velocity equal to the sum of the perturbation velocities induced at that point by the steady flow round the other $n-1$ isolated spheres. This perturbation velocity is, however, no longer given by Stokes flow, but rather by the flow in the Oseen region where inertial effects are as important as viscous effects.

It was seen that for $Re\, s/a \ll 1$, the fluid in the neighbourhood of the cluster was moving upwards relative to the fluid at infinity with speed $\frac{3}{8} n\, Re\, U$. If $Re\, s/a \gg 1$ each sphere is almost entirely outside the perturbation due to the other and each sphere falls faster with speed $U(1-\frac{3}{8}Re)$. For intermediate values more complicated formulae for the speed of fall of the centre of mass are appropriate which are not of primary interest here. However, we may easily compute the rate of expansion of a horizontal plane regular polygon. In this situation the vertical velocities of all the particles are by symmetry the same. From Proudman & Pearson (1957, equation (3.39) at $\mu = 0$) the relative horizontal velocities are given by

$$\frac{d\mathbf{r}_i}{dt} = \frac{3}{2} U \sum_{j \neq i} \left(\frac{Re\, r_{ij}}{a}\right)^{-2} \left\{1 - \left(1 - \frac{1}{2} Re \frac{r_{ij}}{a}\right) \exp - \left(\frac{1}{2} Re \frac{r_{ij}}{a}\right)\right\} \frac{1}{r_{ij}} \mathbf{r}_{ij}.$$

Here $\mathbf{r}_{ij}$ is again the dimensional position vector of the ith vector relative to the jth. As the polygon expands this radial velocity decreases from

$$\tfrac{3}{16} Re\, U \cot(\pi/2n) \quad \text{for} \quad Re\, s/a \ll 1$$

to vary ultimately as $U(Re\, s/a)^{-2}$ for $Re\, s/a \gg 1$. Each individual term in the summation gives a positive contribution to the radial velocity. Hence the rate of expansion is always positive, but ultimately decreases to zero (conclusion (ii)).

The stability of the plane configuration for intermediate values of $Re\, s/a$ presents a difficult problem, but for large values conclusions may be drawn quickly. Each sphere is surrounded at large distances by a perturbation velocity field approximately to a narrow wake behind it (along which there is inflow) and elsewhere to the approximating irrotational radial flow from a virtual source. Provided it does not lie in the wake of any of the others it moves away from them with a velocity which is the sum of terms like

$$\frac{3}{2} U \left(\frac{a}{Re}\right)^2 \frac{1}{r_{ij}^3} \mathbf{r}_{ij},$$

i.e. according to an inverse square law. Thus a plane regular polygon expands and it is easily seen that small departures from the plane configuration are amplified.

This result holds for any value of Re, provided s/a is sufficiently large (conclusion (iv)). The size at which a polygon changes from being stable to unstable cannot be definitely established, but in the unstable region the time scale of the changes in configuration is certainly very much longer than when the polygon is smaller.

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